AMALGAMATED PRODUCTS AND PROPERLY 3-REALIZABLE GROUPS

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Abstract. In this paper, we show that the class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups. We recall that G is said to be properly 3-realizable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a 3-manifold (with boundary).

1. Introduction

We are concerned about the behavior of the property of being properly 3-realizable (for finitely presented groups) with respect to the basic constructions in Combinatorial Group Theory; namely, amalgamated free products and HNN-extensions. Recall that a finitely presented group G is said to be properly 3-realizable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a 3-manifold. It is worth mentioning that the property of being properly 3-realizable has implications in the theory of cohomology of groups, in the sense that if G is properly 3-realizable then for some (equivalently any) compact 2-polyhedron K with $\pi_1(K) \cong G$ we have $H^2_c(\tilde{K};\mathbb{Z})$ free abelian (by manifold duality arguments), and hence so is $H^2(G;\mathbb{Z}G)$ (see [9]). It is a long standing conjecture that $H^2(G;\mathbb{Z}G)$ be free abelian for every finitely presented group G. In [1] it was shown that the property of being properly 3-realizable is preserved under amalgamated free products (HNN-extensions) over finite cyclic groups. See also [3, 4, 7] to learn more about properly 3-realizable groups and related topics. In this paper, we continue in the line of [1]. Our main result is :

Theorem 1.1. *The class of all properly 3-realizable groups is closed under amalgamated free products (and HNN-extensions) over finite groups.*

This generalizes to show that the fundamental group of a finite graph of groups with properly 3-realizable vertex groups and finite edge groups is properly 3-realizable, since such a group can be expressed as a combination of amalgamated free products and HNN-extensions of the vertex groups over the edge groups.

Recall that, given a finitely presented group G and a compact 2-polyhedron K with $\pi_1(K) \cong G$ and $\tilde{K}$ as universal cover, the number of ends of G is the number of ends of $\tilde{K}$ which equals $0, 1, 2$ or ∞ [6] (see also [8, 13]). The 0-ended groups are the finite groups and the 2-ended groups are those having an infinite cyclic subgroup of finite index, and they are all known to be properly 3-realizable (see [1]). Note that Stallings' Structure Theorem [12] characterizes those groups G with more than one end as those which split as an amalgamated free product (or an HNN-extension)

2000 *Mathematics Subject Classification.* Primary 57M07; Secondary 57M10, 57M20.

over a finite group (see also [13, 8]). In addition, Dunwoody [5] showed that this process of further splitting G must terminate after finitely many steps.

Corollary 1.2. *In order to show whether or not all finitely presented groups are properly* 3*-realizable it suffices to look among those groups which are* 1*-ended.*

2. Main result

The purpose of this section is to prove Theorem 1.1. We will make use of the following result :

Proposition 2.1 ([1], Prop. 3.1)**.** *Let M be a manifold of the same proper homotopy type of a locally compact polyhedron K with $dim(K) < dim(M)$. Then, any Freudenthal end $\epsilon \in \mathcal{F}(M)$ can be represented by a sequence of points in ∂M.*

Proof of Theorem 1.1. Let G_0, G_1 be properly 3-realizable groups and F be a finite group with presentation $\langle a_1, \dots, a_N; r_1, \dots, r_M \rangle$. Consider monomorphisms $\varphi_i : F \longrightarrow G_i (i = 0, 1)$, and denote by $G_0 *_F G_1 = \langle G_0, G_1; \varphi_0(a_i) = \varphi_1(a_i), 1 \leq i \leq N \rangle$ the corresponding amalgamated free product. Let X_0, X_1 be compact 2-polyhedra with $\pi_1(X_i) \cong G_i$ and such that their universal covers have the proper homotopy type of 3-manifolds M_0, M_1 respectively. Let $L = \vee_{i=1}^N S^1$ and $f_i : L \longrightarrow X_i$ $(i = 0, 1)$ be cellular maps such that $Im\, f_{i_*} \subseteq \pi_1(X_i)$ corresponds to the subgroup $Im\, \varphi_i \subseteq G_i$. We take the standard 2-dimensional CW-complex Y' associated to the above presentation of F, i.e., Y' has one 1-cell e_i for each generator a_i $(1 \leq i \leq N)$, all of them sharing the only vertex in Y', and one 2-cell d_j for each relation r_j $(1 \leq j \leq M)$ attached via a map $S^1 \longrightarrow \vee_{i=1}^N e_i$ which "'spells" the relation r_j. Consider the adjunction spaces $Y = (\vee_{i=1}^N e_i) \times I \cup_{(\vee_{i=1}^N e_i) \times \{\frac{1}{2}\}} Y'$ (homotopy equivalent to Y') and $Z = Y \cup_{f_0 \times \{0\} \cup f_1 \times \{1\}} (X_0 \sqcup X_1)$. By van Kampen's Theorem, Z is a compact 2-polyhedron with $\pi_1(Z) \cong G_0 *_F G_1$. Let $\tilde{Z}$ be the universal cover of Z with covering map $p : \tilde{Z} \longrightarrow Z$. Then, $p^{-1}(X_i)$ consists of a disjoint union of copies of the universal cover $\tilde{X}_i$ of X_i, since the inclusion $X_i \hookrightarrow Z$ induces a monomorphism $G_i \hookrightarrow G_0 *_F G_1$ between the fundamental groups, $i = 0, 1$ (see [10]). On the other hand, let Γ be a connected component of $p^{-1}(\vee_{i=1}^N e_i) \subset p^{-1}(Y')$ and $\tilde{Y}'$ be the connected component of $p^{-1}(Y')$ containing Γ. Observe that $\tilde{Y}'$ is a copy of the universal cover of Y' (which is compact), as the inclusion $Y' \hookrightarrow Z$ induces a monomorphism $F \hookrightarrow G_0 *_F G_1$. Then, it is easy to see that $p^{-1}(Y)$ consists of a disjoint union of copies of the compact CW-complex $K = (\Gamma \times I) \cup_{\Gamma \times \{\frac{1}{2}\}} \tilde{Y}'$. Thus, $\tilde{Z}$ comes together with the following data (see [13]) :

(a) The disjoint unions $\bigsqcup_{p \in \mathbb{N}} \tilde{X}_{0,p}$ and $\bigsqcup_{r \in \mathbb{N}} \tilde{X}_{1,r}$ of copies of $\tilde{X}_0$ and $\tilde{X}_1$ respectively;

(b) a disjoint union $\bigsqcup_{p,q \in \mathbb{N}} K_{p,q}$ of copies of K; and

(c) a bijective function $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}, (p, q) \mapsto (r, s)$ (given by the group action of $G_0 *_F G_1$ on $\tilde{Z}$), so that for each $p, q \in \mathbb{N}$, $\Gamma \times \{0\} \subset K_{p,q}$ is being glued to $\tilde{X}_{0,p}$ via a lift $\tilde{f}^0_{p,q} : \Gamma \times \{0\} \longrightarrow \tilde{X}_{0,p}$ of the map f_0, and $\Gamma \times \{1\} \subset K_{p,q}$ is being glued to $\tilde{X}_{1,r}$ via a lift $\tilde{f}^1_{r,s} : \Gamma \times \{1\} \longrightarrow \tilde{X}_{1,r}$ of the map f_1.

Next, for each copy of $\tilde{X}_i$, $i = 0, 1$, in $\tilde{Z}$ (written as $\tilde{X}_{0,p}$ or $\tilde{X}_{1,r}$), we take one of the maps $\tilde{f}^i_{\lambda,\mu} : \Gamma \times \{i\} \longrightarrow \tilde{X}_i$ and observe that this map is nullhomotopic so

we can replace it (up to homotopy) with a constant map $g^i_{\lambda,\mu} : \Gamma \times \{i\} \longrightarrow \tilde{X}_i$ with $Im\ g^i_{\lambda,\mu} \subset Im\ \tilde{f}^i_{\lambda,\mu}$, and we do this equivariantly using the group action of G_i on $\tilde{X}_i$. Since this action is properly discontinuous, the collection of all these homotopies gives rise to a proper homotopy equivalence between $\tilde{Z}$ and a new 2-dimensional CW-complex W obtained from a collection of copies of K and a collection of copies of $\tilde{X}_0$ and $\tilde{X}_1$ by gluing each copy of $\Gamma \times \{i\}$ to the corresponding copy of $\tilde{X}_i$ via the bijection φ and the new maps $g^i_{\lambda,\mu}$, $i = 0, 1$.

We will now manipulate the CW-complex K as follows. First, let K' be the CW-complex obtained from K by shrinking to a point $v \times \{i\}$ each copy $T \times \{i\}$ $(i \in I)$ of a maximal tree $T \subset \tilde{Y}' \subset K$. Next, we take K'' to be the CW-complex obtained from K' by identifying the subcomplexes $\Gamma \times \{i\}/T \times \{i\}$, $i = 0, 1$, to a (different) point which we will denote by $[v \times \{0\}]$ and $[v \times \{1\}]$. Note that K'' has a copy of $\tilde{Y}'/T$ as a subcomplex. Since $\tilde{Y}'/T$ is compact and simply connected, it follows from ([14], Prop. 3.3) that $\tilde{Y}'/T$ is homotopy equivalent to a finite bouquet of 2-spheres $\vee_{\alpha \in \mathcal{A}} S^2$ (which we may regard as a connected 2-dimensional CW-complex with no 1-cells). Moreover, we may assume that this homotopy equivalence is given by a cellular map $\tilde{Y}'/T \longrightarrow \vee_{\alpha \in \mathcal{A}} S^2$ so that the 1-skeleton Γ/T of $\tilde{Y}'/T$ is mapped to the wedge point. Finally, taking into account this homotopy equivalence, it is not difficult to see that K'' is homotopy equivalent to the CW-complex $\widehat{K}$ obtained from the disjoint union of a finite bouquet $\vee_{\alpha \in \mathcal{A} \cup \mathcal{B}} S^2$ (where $Card(\mathcal{B}) = 2\ rank(\pi_1(\Gamma))$ and the unit interval I by identifying $\frac{1}{2} \in I$ with the wedge point, so that $I \subset \widehat{K}$ would correspond to the subcomplex $v \times I \subset K'$ and $0, 1 \in I$ would correspond to $[v \times \{0\}], [v \times \{1\}] \in K''$. Notice that $\widehat{K}$ thickens to a 3-manifold $P \searrow \widehat{K}$ containing 3-dimensional 1-handles H and H' (with a free end face each of them) corresponding to the edges $[0, \frac{1}{2}], [\frac{1}{2}, 1] \subset I \subset \widehat{K}$ respectively.

According to the above, one can see that the CW-complex W (proper homotopy equivalent to $\tilde{Z}$) is in turn proper homotopy equivalent to the quotient space obtained from the following data :

(a) A disjoint union $\bigsqcup\limits_{p \in \mathbb{N}} \tilde{X}_{0,p}$ of copies of $\tilde{X}_0$ together with a locally finite sequence of points $\{x^p_q\}_{q \in \mathbb{N}} \subset \tilde{X}_{0,p}$, for each $p \in \mathbb{N}$, corresponding to the images of the constant maps $g^0_{p,q} : \Gamma \times \{0\} \longrightarrow \tilde{X}_{0,p}$ considered above in the construction of W;

(b) a disjoint union $\bigsqcup\limits_{r \in \mathbb{N}} \tilde{X}_{1,r}$ of copies of $\tilde{X}_1$ together with a locally finite sequence of points $\{y^r_s\}_{s \in \mathbb{N}} \subset \tilde{X}_{1,r}$, for each $r \in \mathbb{N}$, corresponding to the images of the constant maps $g^1_{r,s} : \Gamma \times \{1\} \longrightarrow \tilde{X}_{1,r}$ from the construction of W;

(c) a disjoint union $\bigsqcup\limits_{p,q \in \mathbb{N}} \widehat{K}_{p,q}$ of copies of $\widehat{K}$; and

(d) the bijective function $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}, (p, q) \mapsto (r, s)$, so that $0 \in I \subset \widehat{K}_{p,q}$ is being identified with $x^p_q \in \tilde{X}_{0,p}$ and $1 \in I \subset \widehat{K}_{p,q}$ is being identified with $y^r_s \in \tilde{X}_{1,r}$ $((r, s) = \varphi(p, q))$, for each $p, q \in \mathbb{N}$.

We now follow an argument similar to the proof of ([1], Lemma 3.2). Fix proper homotopy equivalences $h : \tilde{X}_0 \longrightarrow M$ and $h' : \tilde{X}_1 \longrightarrow N$, where we now denote M_0 by M and M_1 by N. Given the above data, we set $A = \mathbb{N} \times \mathbb{N}$ and consider

maps $i : A \longrightarrow \bigsqcup_{p\in\mathbb{N}} \tilde{X}_{0,p}$, $i' : A \longrightarrow \bigsqcup_{r\in\mathbb{N}} \tilde{X}_{1,r}$ given by $i(p,q) = x^p_q$ and $i'(p,q) = y^r_s$, where $(r,s) = \varphi(p,q)$. It is easy to check that i and i' are proper cofibrations, as the corresponding sequences of points are locally finite. Next, we take exhaustive sequences $\{A^p_m\}_{m\in\mathbb{N}}$ and $\{B^r_n\}_{n\in\mathbb{N}}$ of copies M_p and N_r of the 3-manifolds M and N respectively by compact submanifolds, and define proper cofibrations $j : A \longrightarrow \bigsqcup_{p\in\mathbb{N}} M_p$, $j' : A \longrightarrow \bigsqcup_{r\in\mathbb{N}} N_r$ as follows. Given $(p,q) \in A$ and the proper homotopy equivalences $h_p = h : \tilde{X}_{0,p} \longrightarrow M_p$, $h'_r = h' : \tilde{X}_{1,r} \longrightarrow N_r$ (with $(r,s) = \varphi(p,q)$), we take $m(q), n(s) \in \mathbb{N}$ to be the least natural numbers such that $h_p \circ i(p,q) \notin A^p_{m(q)} \subset M_p$ and $h'_r \circ i'(p,q) \notin B^r_{n(s)} \subset N_r$. Then, using Proposition 2.1, we define $j(p,q)$ and $j'(p,q)$ to be points $j(p,q) = a_{p,q} \in \partial M_p - A^p_{m(q)}$ and $j'(p,q) = b_{r,s} \in \partial N_r - B^r_{n(s)}$ so that (i) j, j' are one-to-one maps (note that h, h' need not be one-to-one); and (ii) $a_{p,q}$ and $h_p \circ i(p,q)$ (resp. $b_{r,s}$ and $h'_r \circ i'(p,q)$) are in the same path component of $M_p - A^p_{m(q)}$ (resp. $N_r - B^r_{n(s)}$). Notice that j and j' are proper maps by construction. Consider now maps

$$G : \left(\bigsqcup_{p\in\mathbb{N}} \tilde{X}_{0,p}\right) \times \{0\} \cup (i(A) \times I) \longrightarrow \bigsqcup_{p\in\mathbb{N}} M_p$$

$$H : \left(\bigsqcup_{r\in\mathbb{N}} \tilde{X}_{1,r}\right) \times \{0\} \cup (i'(A) \times I) \longrightarrow \bigsqcup_{r\in\mathbb{N}} N_r$$

with $G|_{\tilde{X}_{0,p}\times\{0\}} = h_p = h$ and $H|_{\tilde{X}_{1,r}\times\{0\}} = h'_r = h'$ ($p, r \in \mathbb{N}$), and so that $\alpha_{p,q} = G|_{i(p,q)\times I}$ (resp. $\beta_{r,s} = H|_{i'(p,q)\times I}$) is a path in $M_p - A^p_{m(q)}$ from $h_p \circ i(p,q)$ to $a_{p,q}$ (resp. a path in $N_r - B^r_{n(s)}$ from $h'_r \circ i'(p,q)$ to $b_{r,s}$). Observe that G and H are proper maps, since h, h', j and j' are proper. By the Homotopy Extension Property, the maps G, H extend to proper maps

$$\widehat{G} : \left(\bigsqcup_{p\in\mathbb{N}} \tilde{X}_{0,p}\right) \times I \longrightarrow \bigsqcup_{p\in\mathbb{N}} M_p \ \ , \ \ \widehat{H} : \left(\bigsqcup_{r\in\mathbb{N}} \tilde{X}_{1,r}\right) \times I \longrightarrow \bigsqcup_{r\in\mathbb{N}} N_r$$

which yield commutative diagrams

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ \bigsqcup_{p\in\mathbb{N}} \tilde{X}_{0,p} \arrow[rr, "\hat{h}"] & & \bigsqcup_{p\in\mathbb{N}} M_p \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "i'"'] \arrow[dr, "j'"] & \\ \bigsqcup_{r\in\mathbb{N}} \tilde{X}_{1,r} \arrow[rr, "\hat{h}'"] & & \bigsqcup_{r\in\mathbb{N}} N_r \end{tikzcd}$$

where $\hat{h} = \widehat{G}|_{(\bigsqcup_{p\in\mathbb{N}} \tilde{X}_{0,p})\times\{1\}}$ and $\hat{h}' = \widehat{H}|_{(\bigsqcup_{r\in\mathbb{N}} \tilde{X}_{1,r})\times\{1\}}$ are proper homotopy equivalences. Moreover, $\hat{h}$ and $\hat{h}'$ are proper homotopy equivalences under A, by ([2], Prop. 4.16) (compare with [11], Chap. 6, § 5). Hence, they induce a proper homotopy equivalence between the quotient space described above (proper homotopy equivalent to W) and the following 3-manifold obtained as the quotient space given by the data :

(a) The disjoint unions $\bigsqcup_{p\in\mathbb{N}} M_p$ and $\bigsqcup_{r\in\mathbb{N}} N_r$ of copies of the 3-manifolds M and N respectively;

(b) a disjoint union $\bigsqcup_{p,q\in\mathbb{N}} P_{p,q}$ of copies of the compact 3-manifold $P \searrow \widehat{K}$; and

(c) the bijective function $\varphi : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}, (p,q) \mapsto (r,s)$, so that for each $p,q \in \mathbb{N}$, the free ends of the corresponding 3-dimensional 1-handles $H_{p,q}, H'_{p,q} \subset P_{p,q}$ considered above are being identified homeomorphically with small disks $D_{p,q} \subset \partial M_p$ and $D'_{r,s} \subset \partial N_r$ about the points $a_{p,q}$ and $b_{r,s}$ respectively.

In the case of an HNN-extension $G*_F = \langle G, t; t^{-1}\psi_0(a_i)t = \psi_1(a_i), 1 \leq i \leq N\rangle$ (with monomorphisms $\psi_i : F \longrightarrow G, i = 0,1$), let X be a compact 2-polyhedron with $\pi_1(X) \cong G$ and whose universal cover has the proper homotopy type of a 3-manifold, and let $f_i : \vee_{i=1}^N S^1 \longrightarrow X$ $(i = 0,1)$ be cellular maps so that $Im\ f_{i_*} \subseteq \pi_1(X)$ corresponds to the subgroup $Im\ \psi_i \subseteq G$. Let Y be the 2-dimensional CW-complex constructed as above and consider the adjunction space $Z = Y \cup_{f_0\times\{0\}\cup f_1\times\{1\}} X$, with $\pi_1(Z) \cong G*_F$. Then, the proof goes just as the one given above for the amalgamated free product. □

Acknowledgements

The first three authors were supported by the project MTM 2004-01865. This research was also supported by Slovenian-Spanish research grant BI-ES/04-05-014.

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